

THE FORMULATION OF D -AFFINE CONNECTION

TAKUYA SAEKI

D -擬似接続の構成

佐 伯 卓 也

Recently H. Flanders [2, 3]¹⁾ has defined the affine connection by extending the exterior differentiation operator to the operator that acts on some linear spaces in place of exterior differential forms. It is known that there exists a divergence operator of skew-symmetric tensors on an orientable C^∞ -manifold M , and it is useful to define the homology group on M if M is paracompact. On the other hand, the exterior differentiation operator plays the important rôle of defining the de Rham cohomology group on M . If M is orientable and paracompact, these groups are isomorphic. Like this the exterior differentiation operator and the divergence operator are mutually dual. So we arrive at the question if we can define an analogous notion of the affine connection by employing the divergence operator on M . The solution of the question is the purpose of the present paper.

1. **Preliminaries.** Let M be a connected C^∞ -manifold and \mathfrak{F} the ring of all C^∞ -functions on M . For a point $x \in M$ we denote by $\Lambda(x) = \Sigma \oplus \Lambda^p(x)$ (direct-some) the graded algebra of p -tensors, by convention, $\Lambda^1(x) =$ the tangent space of M at x and $\Lambda^0(x) = R$ (the field of real numbers). If we put $\Lambda(M) = \bigcup_{x \in M} \Lambda(x)$, this is a fibre bundle over M . Let v be a C^∞ -section of the bundle. Hence $v(x) \in \Lambda(x)$. If, for all $x \in M$, $v(x) \in \Lambda^p(x)$ for some fixed p , v is called a p -tensor field. We denote by T_p the space of p -tensor fields with $T_1 =$ the space of tangent vector fields and $T_0 = \mathfrak{F}$. If we put $T = \Sigma \oplus T_p$, T is a graded algebra over \mathfrak{F} .

As before, for a point $x \in M$ we denote by $\Lambda^*(x) = \Sigma \oplus \Lambda^{*q}(x)$ the graded algebra of exterior forms on the tangent space at x . By convention $\Lambda^{*1}(x) = T_x^*(M)$ (the dual space of $T_x(M)$) and $\Lambda^{*0}(x) = R$. If we put $\Lambda^*(M) = \bigcup_{x \in M} \Lambda^*(x)$, this is a fibre bundle over M , and its C^∞ -section ω is called an *exterior polynomial*. Hence we have $\omega(x) \in \Lambda^*(x)$. If, for all $x \in M$, $\omega(x) \in \Lambda^{*q}(x)$ for some fixed q , ω is called an *exterior differential form of degree q* , or simply a q -form. We denote by T^q the space of q -forms with $T^1 =$ the space of tangent covector fields and $T^0 = \mathfrak{F}$. If we put $T^* = \Sigma \oplus T^q$, T^* is a graded algebra over \mathfrak{F} .

We shall consider a tensor products

$$T_p^q = T_p \otimes T^q.$$

1) Numbers in brackets refer to the bibliography at the end of the paper.

These spaces may be considered from two points of view: either as tensor products of the spaces T_p and T^q over the ring \mathfrak{F} , or as sections of the fibre bundle of all the elements of all $(\Lambda^p(x)) \otimes (\Lambda^{*q}(x))$, where the latter tensor product is taken over R . This implies, passing to homogeneous components, the existence of an operator on $T_p^q \times T_{p'}^{q'}$ to $T_{p+p'}^{q+q'}$ given by the linearity and

$$(1. 1) \quad (v \wedge w) \otimes (\omega \wedge \eta) = (v \otimes \omega) \wedge (w \otimes \eta)$$

where $v \in T_p$, $w \in T_{p'}$, $\omega \in T^q$ and $\eta \in T^{q'}$. It is easy to see that this operator is associative and distributive, and obeys the following commutation rule

$$(1. 2) \quad X \wedge Y = (-1)^{pp'+qq'} Y \wedge X.$$

where $X \in T_p^q$ and $Y \in T_{p'}^{q'}$. Clearly $\Sigma \oplus T_p^q$ is a graded algebra over \mathfrak{F} and naturally identified with $T \otimes T^*$.

Now we assume that M is orientable. Let ρ be a positive scalar density on M . In a coordinate neighbourhood U with a local coordinate system x^1, \dots, x^n , a p -tensor v can be written by

$$(1. 3) \quad v = v^{i_1 \dots i_p}(x^1, \dots, x^n) e_{i_1} \wedge \dots \wedge e_{i_p} \quad (i_1, \dots, i_p = 1, \dots, n)$$

where $e_{i_k} = \partial/\partial x^{i_k}$ ($k = 1, \dots, p$). If $v \in T_p$ ($p \geq 1$) we define an $(n-p)$ -form $*v$ by

$$(1. 4) \quad *v = \rho v^{i_1 \dots i_p} \varepsilon_{i_1 \dots i_p j_1 \dots j_{n-p}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}},$$

where $\varepsilon_{i_1 \dots i_p j_1 \dots j_{n-p}}$ is a component of the Kronecker's tensor. The operator $*$ is \mathfrak{F} -linear and has an inverse.

For a q -form ω ($q \geq 1$) written by

$$(1. 5) \quad \omega = a_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q},$$

another \mathfrak{F} -linear operator $*'$ is defined by

$$(1. 6) \quad *'\omega = (1/\rho) a_{i_1 \dots i_q} \varepsilon^{i_1 \dots i_q j_1 \dots j_{n-q}} e_{j_1} \wedge \dots \wedge e_{j_{n-q}}.$$

By convention for $p=0$ and $q=0$, $*1 = \rho dx^1 \wedge \dots \wedge dx^n$ and $*'1 = (1/\rho) e_1 \wedge \dots \wedge e_n$ respectively.

Again consider a p -tensor field, we operate $*$ and $*'$ for v by succession, and we have

$$(*'*)v = *'(*v) = (-1)^{p(n-p)} v.$$

Similarly we have for a q -form

$$(*'*)\omega = *'(*'\omega) = (-1)^{q(n-q)} \omega.$$

Hence we have

$$(1. 7) \quad *v = (-1)^{p(n-p)} *'^{-1} v, \quad *'\omega = (-1)^{q(n-q)} *^{-1} \omega.$$

2. The dual wedge product In § 3, we shall define the divergence of a p -tensor field. A usual wedge product of two divergence-free p -tensor fields is not in general divergence-free. Let us now construct a new and naturally defined product of two divergence-free p -tensor fields which is divergence-free. This product will be called

a *dual wedge product*.

For a p -tensor field v and p' -tensor field w , the dual wedge product is defined by

$$(2. 1) \quad v \circ w = \begin{cases} *^{-1}(*v \wedge *w) & \text{for } p + p' \geq n, \\ 0 & \text{for } p + p' < n. \end{cases}$$

Similarly for q -form ω and q' -form η , the dual wedge product is defined by

$$(2. 1') \quad \omega \circ \eta = \begin{cases} *'^{-1}(*'\omega \wedge *'\eta) & \text{for } q + q' \geq n, \\ 0 & \text{for } q + q' < n. \end{cases}$$

It is clear that these products are associative and distributive.

Next we shall define the dual wedge product for the space $\Sigma \oplus T_p^q$, in fact, there exists an operator on $T_p^q \times T_{p'}^{q'}$ to $T_{n-(p+p')}^{n-(q+q')}$ given by the linearity and

$$(2. 2) \quad (v \circ w) \otimes (w \circ \eta) = (v \otimes \omega) \circ (w \otimes \eta),$$

where $v \in T_p$, $w \in T_{p'}$, $\omega \in T^q$ and $\eta \in T^{q'}$. This operator is also associative and distributive, and obeys the following commutation rule

$$(2. 3) \quad X \circ Y = (-1)^{(n-p)(n-p')+(n-q)(n-q')} Y \circ X,$$

where $X \in T_p^q$ and $Y \in T_{p'}^{q'}$.

By convention we shall introduce a \mathfrak{F} -linear operator $\#$ by

$$(2. 4) \quad \#X = (* \otimes *)X = (*v) \otimes (*'\omega),$$

where $X = v \otimes \omega \in T_p^q$. Hence we have

$$X \circ Y = \#^{-1}(\#X \wedge \#Y),$$

where $X \in T_p^q$ and $Y \in T_{p'}^{q'}$.

3. The divergence of p -tensor fields. Let M be a connected n -dimensional orientable C^∞ -manifold and v a p -tensor field on M . Writing locally v by (1. 3), we define the divergence of a p -tensor v , written by ∂v , by

$$(3. 1) \quad \partial v = \frac{1}{\rho} \frac{\partial(\rho v^{i_1 \dots i_p})}{\partial x^j} e_{i_1} \wedge \dots \wedge e_{i_p},$$

where ρ is a positive scalar density on M . An easy calculation shows $\partial^2 = \partial \partial = 0$.

PROPOSITION 3.1. *In the graded algebra $T = \Sigma \oplus T^p$ over \mathfrak{F} (see § 1), there exists a unique operator ∂ , called the divergence, satisfying the following conditions:*

- (1) ∂ is R -linear,
- (2) $v \in T_p$ implies $\partial v \in T_{p-1}$,
- (3) $\partial(v \circ w) = \partial v \circ w + (-1)^{n-p} v \circ \partial w$ for $v \in T_p$,
- (4) $\partial^2 v = 0$.

In fact, for the operators $*$ and $*'$ defined in § 1, we have

$$(3. 2) \quad d* = *d \text{ and } \partial*' = (-1)^{q(n-q)} *d$$

for a p -tensor field and for a q -form respectively. Hence for $v \in T_p$ and $w \in T_q$, by

employing (3.2), we have

$$\begin{aligned}
 \partial(v \circ w) &= \partial\{*^{-1}(*v \wedge *w)\} \\
 &= *^{-1}d(*v \wedge *w) \\
 &= *^{-1}\{d*v \wedge *w + (-1)^{n-p}*v \wedge d*w\} \\
 &= *^{-1}\{*(\partial v) \wedge *w\} + (-1)^{n-p}*^{-1}\{*v \wedge *(\partial w)\} \\
 &= \partial v \circ w + (-1)^{n-p}v \circ \partial w
 \end{aligned}$$

which shows (3) in Proposition 3.1.

A p -tensor field v is called *divergence-free*, if $\partial v = 0$. (3) of Proposition 3.1 shows that the dual wedge product of two divergence-free tensor fields is divergence-free. But we can not expect this fact for the usual wedge product of p -tensor fields. [4]

Define the linear space of divergence-free p -tensors by

$$Z_p(T, R) = \{v \in T_p \mid \partial v = 0\}$$

and the linear space of p -tensor field divergences by

$$B_p(T, R) = \{v \in T_p \mid v = \partial w \text{ for some } w \in T_{p+1}\}.$$

We shall call $H_p(T, R)$ by the *de Rham homology group*, where $H_p(T, R)$ is a real coefficient group on M defined by $Z_p(T, R)/B_p(T, R)$. (3.2) asserts that, for p -tensor field v , $*v$ is closed if and only if $v \in Z_p(T, R)$, and $*v$ is exact if and only if $v \in B_p(T, R)$. Then the operator $*$ induces an isomorphism between $H_p(T, R)$ and the $(n-p)$ th de Rham cohomology group $H^{n-p}(T^*, R)$ on M . If M is compact, this result is nothing but the Poincaré duality theorem.

It is known that, if M is a connected orientable paracompact C^∞ -manifold, the $(n-p)$ th de Rham cohomology group $H^{n-p}(T^*, R)$ on M is isomorphic onto the p -th real coefficient singular homology group $H_p(S_M)$. On the other hand, it is known that if M is paracompact $H_p(S_M)$ is isomorphic onto $H_p(M, R) = \varprojlim H_p(\mathcal{U}, R)$, where $H_p(\mathcal{U}, R)$ is the real coefficient homology group of a covering \mathcal{U} of M and \varprojlim means the projective limit for coverings of M . The group $H_p(M, R)$ is called a *p -th real coefficient homology group of M* .

PROPOSITION 3.2. *Let M be an orientable connected paracompact C^∞ -manifold of the dimension n . The p -th de Rham homology group $H_p(T, R)$ on M is isomorphic with the p -th real coefficient singular homology group $H_p(S_M)$ on M and the p -th real coefficient homology group of M , i. e., $H_p(M, R)$.*

4. The D -affine connection. In the previous section, we have defined the divergence of the p -tensor field. In this section, we shall consider the question about the extension of the divergence from on p -tensor fields to on the space T_p^q . For the purpose we shall define the D -affine connection which is considered as a dual of the usual affine connection in a certain sense. Let M be an orientable C^∞ -manifold.

DEFINITION 4.1. A D -affine connection on M is an operator ∂ on T_n^{n-1} into T_{n-1}^{n-1} satisfying the following two properties;

- (1) $\partial(X + Y) = \partial X + \partial Y$,
- (2) $\partial(aX) = (*'da) \circ X + a\partial X$,

where $a \in \mathfrak{F}$ and d is the exterior differentiation operator.

We shall construct the unique operator on each T_p^q to T_{p-1}^q by virtue of the operator ∂ which defines the D -affine connection.

Now let us consider the mapping

$$(\#X_1, \dots, \#X_{n-q}) \rightarrow \sum_{j=1}^{n-q} \#X_1 \wedge \dots \wedge \#X_{j-1} \wedge \#(\partial X_j) \wedge \#X_{j+1} \wedge \dots \wedge \#X_{n-q},$$

where $X_1, \dots, X_{n-q} \in T_n^{n-1}$, hence the mapping is on $(n-q)$ -tuples of elements of T_1^0 into T_{n-q}^1 . This is alternating and multilinear, hence there exists a linear mapping f on T_{n-q}^0 to T_{n-q}^1 such that

$$f(\#X_1 \wedge \dots \wedge \#X_{n-q}) = \sum_{j=1}^{n-q} \#X_1 \wedge \dots \wedge \#X_{j-1} \wedge \#(\partial X_j) \wedge \#X_{j+1} \wedge \dots \wedge \#X_{n-q}$$

for any elements X_1, \dots, X_{n-q} in $T_n^{n-1}[1]$. By employing (2. 5), we have

$$\#X_1 \wedge \dots \wedge \#X_{n-q} = \#(X_1 \circ \dots \circ X_{n-q})$$

and

$$\begin{aligned} \sum_{j=1}^{n-q} \#X_1 \wedge \dots \wedge \#X_{j-1} \wedge \#(\partial X_j) \wedge \#X_{j+1} \wedge \dots \wedge \#X_{n-q} \\ = \# \sum_{j=1}^{n-q} X_1 \circ \dots \circ X_{j-1} \circ \partial X_j \circ X_{j+1} \circ \dots \circ X_{n-q}. \end{aligned}$$

Hence we have

$$\#^{-1}f\#(X_1 \circ \dots \circ X_{n-q}) = \sum_{j=1}^{n-q} X_1 \circ \dots \circ X_{j-1} \circ \partial X_j \circ X_{j+1} \circ \dots \circ X_{n-q}.$$

Putting $\partial = \#^{-1}f\#$ on T_n^q , we have

$$(4. 1) \quad \partial(X_1 \circ \dots \circ X_{n-q}) = \sum_{j=1}^{n-q} X_1 \circ \dots \circ X_{j-1} \circ \partial X_j \circ X_{j+1} \circ \dots \circ X_{n-q},$$

where $X_1 \circ \dots \circ X_{n-q} \in T_n^q$ and the right hand side of (4. 1) belongs to T_{n-1}^q . Then we have a linear mapping ∂ on T_n^q into T_{n-1}^q

Consider $X \in T_p^q$, by linearity, it suffices to consider the case $X = v \circ \omega$, where $v \in T_p$ and $\omega \in T^q$. Uniquely we can put

$$(4. 2) \quad X = v \circ \omega = v \circ X'$$

as a element of $T_p \circ T_n^q$, where $X' = *'1 \circ \omega$.

Consider the mapping

$$(\#v, \#X') \rightarrow \#(\partial v) \wedge \#X' + (-1)^{n-p} \#v \wedge \#(\partial X'),$$

where $v \in T_p$ and $X' \in T_n^q$. This mapping is bilinear, hence there exists a linear mapping f' on $\#T_p \otimes \#T_n^q$ into T_{n-p}^{n-q+1} such that

$$f'(\#v \wedge \#X') = \#(\partial v) \wedge \#X' + (-1)^{n-q} \#v \wedge \#(\partial X')$$

for any $v \in T_p$ and any $X' \in T_n^q$ [1]. By (2. 5), we have

$$\#v \wedge \#X' = \#(v \circ X')$$

and

$$\#(\partial v) \wedge \#X' + (-1)^{n-p} \#v \wedge \#(\partial X') = \#\{(\partial v \circ X') + (-1)^{n-p} (v \circ \partial X')\}.$$

Hence we have

$$\#^{-1} f' \#(v \circ X') = \partial v \circ X' + (-1)^{n-q} (v \circ \partial X').$$

Putting $\partial = \#^{-1} f' \#$ on T_p^1 , we can conclude

$$(4. 3) \quad \partial(v \circ X') = \partial v \circ X' + (-1)^{n-p} (v \circ \partial X'),$$

where $X = v \circ X' \in T_p^q$ and the right hand side of (4. 3) belongs to T_{p-1}^q . Thus there exists a linear mapping ∂ on T_p^q into T_{p-1}^q . We can easily verify the uniqueness of such ∂ .

PROPOSITION 4.2. *If ∂ is a D-affine connection on M , then there exists a unique collection of operators, also denoted by ∂ , one on each space T_p^q to T_{p-1}^q ($q < n$) and satisfying the following properties :*

- (1) $\partial(X + Y) = \partial X + \partial Y$ for $X \in T_p^q$, and $Y \in T_p^q$,
- (2) $\partial(X \circ Y) = \partial X \circ Y + (-1)^{n-p} X \circ \partial Y$ for $X \in T_p^q$, $Y \in T_{p'}^q$,
- (3) ∂ coincides with the given D-affine connection on T_n^{n-1} , and coincides with the divergence on $T_n^0 = T_p$.

For $X \in T_p^n$ ($p = 1, \dots, n$) we define

$$(4. 4) \quad \partial X = *d\#X,$$

where d is the exterior differentiation operator. Hence $\partial^2 X = 0$, if $X \in T_p^n$ ($p = 1, \dots, n$).

To prove the Proposition 4.2, the proof of (2) remains. By linearity it suffices to prove this in the following special case: $X = v \circ X'$ and $Y = w \circ Y'$, where we take the identification (4. 2) and $X \in T_p^q$, $Y \in T_{p'}^q$ and $v \in T_p$, $w \in T_{p'}$, $X' \in T_n^q$, $Y' \in T_n^q$. Hence

$$\begin{aligned} \partial(X \circ Y) &= \partial\{(v \circ X') \circ (w \circ Y')\} \\ &= (-1)^{n(n-q)} \partial\{(v \circ w) \circ (X' \circ Y')\} \\ &= (-1)^{n(n-q)} \{\partial(v \circ w) \circ (X' \circ Y') + (-1)^{n-(p+p'-n)} (v \circ w) \circ \partial(X' \circ Y')\} \\ &= (-1)^{n(n-q)} \{(\partial v \circ w) + (-1)^{n-p} v \circ \partial w\} \circ (X' \circ Y') \\ &\quad + (-1)^{(n-p)+(n-p')}(v \circ w) \circ (\partial X' \circ Y' + X' \circ \partial Y') \\ &= (-1)^{n(n-q)} \{(\partial v \circ w) \circ (X' \circ Y') + (-1)^{n-p} (v \circ \partial w) \circ (X' \circ Y') \\ &\quad + (-1)^{(n-p)+(n-p')} [(v \circ w) \circ (\partial X' \circ Y') + (v \circ w) \circ (X' \circ \partial Y')]\} \end{aligned}$$

$$\begin{aligned}
&= (\partial v \circ X') \circ (w \circ Y') + (-1)^{n-p} (v \circ X') \circ (\partial w \circ Y') \\
&\quad + (-1)^{n-p-p'+(n-p')+n} (v \circ \partial X') \circ (w \circ Y') + (-1)^{(n-p)+(n-p')} (v \circ X') \circ (w \circ \partial Y') \\
&= \partial(v \circ X') \circ (w \circ Y') + (-1)^{n-p} (v \circ X') \circ \partial(w \circ Y') \\
&= \partial X \circ Y + (-1)^{n-p} X \circ \partial Y.
\end{aligned}$$

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MATHEMATICAL INSTITUTE,
 IWATE UNIVERSITY,
 MORIOKA.